Adiabatic Invariants

As we know in classical mechanics that whenever a system has periodic motion, the action integral $\oint p\,dq$ taken over a period is constant of motion where p and q are generalised momentum and co-ordinate which repeat themselves in the motion. If a slow change is made in the system, so that the motion is not quite periodic, the constant of motion does not change and is then called an adiabatic invariant. By slow here we mean slow compared with the period of motion. Adiabatic invariants play an important role in plasma physics. There are three types of Adiabatic invariants, each corresponding to a different type of periodic motion.

1. The First Adiabatic Invariant, μ

As we already know that Magnetic Moment

$$\mu = \frac{1}{2} m v_\perp^2 / B$$

Magnetic moment of a current loop with area A and current I is $\mu = IA$

In case of singly charged ion, I is generated by charge e comming around $\omega_c/2\pi$ times a second

$$\therefore I = \frac{e\omega_c}{2\pi}, \quad A = \pi r_L^2 = \pi v_\perp^2/\omega_c^2$$

$$\text{Thus } \mu = \frac{\pi v_\perp^2}{\omega_c^2}\frac{e\omega_c}{2\pi} = \frac{1}{2} v_\perp^2 e/\omega_c = \frac{1}{2} m v_\perp^2/B$$

As the particle moves into regions of stronger to weaker B, its Larmor radius changes but μ remains invariant. To see this, let us consider

$$m\frac{dv_{\parallel}}{dt} = -\mu\frac{\partial B}{\partial s}$$

or $$m v_{\parallel}\frac{dv_{\parallel}}{dt} = \frac{d}{dt}\left(\frac{1}{2}m v_{\parallel}^2\right) = -\mu\frac{\partial B}{\partial s}\frac{\partial s}{\partial t}$$

$$= -\mu\frac{dB}{dt}$$

Here, the Particle's energy must be Conserved, So we have

$$\frac{d}{dt}\left(\frac{1}{2}m v_{\parallel}^2 + \frac{1}{2}m v_{\perp}^2\right) = \frac{d}{dt}\left(\frac{1}{2}m v_{\parallel}^2 + \mu B\right) = 0$$

$$\cancel{\therefore \mu\frac{dB}{dt} = \frac{d}{dt}(\mu B)}$$

It follows that $$\frac{d}{dt}(\mu B) = -\frac{d}{dt}\left(\frac{1}{2}m v_{\parallel}^2\right)$$

$$= \mu\frac{dB}{dt}$$

$$\therefore -\mu\frac{dB}{dt} - \frac{d}{dt}(\mu B) = 0$$

So $$\frac{d\mu}{dt} = 0$$

Hence μ is a Constant.

Which is the first adiabatic invariant.

<u>Alternate Proof</u> — As earlier we met $\mu = \frac{1}{2}m v_{\perp}^2/B$

If we take p to be angular momentum $m v_{\perp} r$ and dq to be the Co-ordinate $d\theta$, then the action integral becomes

$$\oint p\,dq = \oint m v_{\perp} r_L\,d\theta = 2\pi r_L m v_{\perp} = 2\pi\frac{m v_{\perp}^2}{\omega_c} = 4\pi\frac{m}{|q|}\mu$$

Thus μ is a Constant of the motion as long as q/m is not changed

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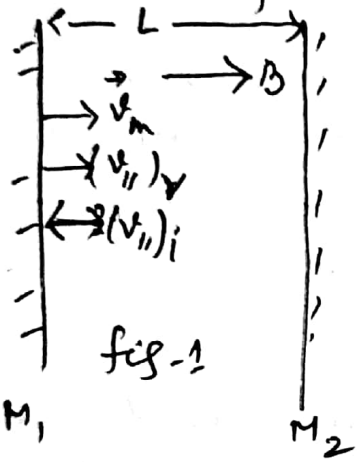
The Second Adiabatic Invariant, J

Let us consider a Particle trapped between two magnetic mirrors. It bounces between them and therefore has a periodic motion at the bounce frequency as the seperation distance between two mirrors changes very slowly. With the Periodic motion of the Particles between two magnetic mirrors (whose seperation varies slowly in time) there is associated an adiabatic invariant called the Longitudinal Adiabatic Invariant, defined by integral $J = \oint \vec{V}\cdot d\vec{l} = \oint v_{\parallel}\, dl$ — (1) taken over one period of oscillation of the particle back and forth between the mirror points.



For its simple Proof of Adiabatic invariance J (i.e. Second Adiabatic invariance) let us consider fig 1 where existing $\vec{B}$ field in the z-direction is uniform in space except near the Points M_1 & M_2, where the field increases to form the mirrors seperated by a distance L. suppose that Mirror M_1 approaches the other one with velocity $\vec{v}_m = -\frac{d\vec{L}}{dt}$ — (2) the -ve sign being due to the fact that L decreases with time. It is assumed that $\vec{v}_m \ll v_{\parallel}$. Thus, the distance moved by Mirror M_1 during one period of oscillation is very small compared to the distance L between Mirrors.

Further, since $\vec{B}$ is assumed to be uniform throughout the space (except near the ends), the Longitudinal Particle speed $v_{\parallel}$ may be taken to be constant. Neglecting the small end effects at the two mirrors we can take $J = \int_0^{2L} v_{\parallel}\, dl = 2v_{\parallel} L$ — (3)

The time rate of change of $\vec{J}$ is

$$\frac{d\vec{J}}{dt} = 2v_{\parallel}\frac{d\vec{L}}{dt} + 2L\frac{dv_{\parallel}}{dt} = -2v_{\parallel}v_m + 2L\frac{dv_{\parallel}}{dt} \quad (4)$$

where eqn (2) has been used. To Calculate $dv_{\parallel}/dt$ we set

$$\frac{dv_{\parallel}}{dt} = \frac{\Delta v_{\parallel}}{\Delta t} = \frac{\Delta v_{\parallel}}{2L/v_{\parallel}} \quad (5)$$

Where $\Delta v_{\parallel}$ denotes the change in the Particle speed $v_{\parallel}$ on reflection from the moving mirror, and $\Delta t = 2L/v_{\parallel}$ is the period of oscillation between two mirrors. In order to find $\Delta v_{\parallel}$ it is convenient to transform to a coordinate system moving with the magnetic mirror M_1 at the speed v_m. Let us denote this moving co-ordinate system by a prime and the incident and reflected Particle speeds by subscripts i and r, respectively. Thus

$$(v_{\parallel})'_i = (v_{\parallel})_i + v_m \quad (6)$$

$$(v_{\parallel})'_r = (v_{\parallel})_r - v_m \quad (7)$$

which gives the change in particle speed in one reflection

$$\Delta v_{\parallel} = (v_{\parallel})_r - (v_{\parallel})_i = 2v_m \quad (8)$$

Since in the moving co-ordinate $(v_{\parallel})'_i = (v_{\parallel})'_r$ with only their direction reversed. Therefore eqn (5) becomes

$$\frac{dv_{\parallel}}{dt} = \frac{2v_m}{2L/v_{\parallel}} = \frac{v_m v_{\parallel}}{L} \quad (9)$$

Putting this result into (4), we get

$$\frac{d\vec{J}}{dt} = \frac{d}{dt}(2v_{\parallel}\vec{L}) = 0 \quad (10)$$

Which shows that J = Const i.e. an Adiabatic Invariant. Thus the second Adiabatic invariant is proved.

The Third Adiabatic Invariant, ϕ

The third adiabatic invariant is related to the constancy of the magnetic flux through the drift orbits. It is given by

$$\phi = \int_S \vec{B} \cdot d\vec{s}$$ where s is the area enclosed by the particles drift orbit.

$$= \pi r_c^2 B = \pi \frac{m^2 v_\perp^2}{q^2 B^2} B = \frac{2\pi m}{q^2}\left(\frac{1}{2} m v_\perp^2 / B\right)$$

Therefore $$\frac{d}{dt}(\phi) = \frac{2\pi m}{q^2} |\mu| = 0$$

in view of the invariance of $|\mu|$. Hence, as the charged particle moves in a region of converging $\vec{B}$ field, it will orbit with increasingly smaller radius, so that the magnetic flux enclosed by the orbit remains constant.