Several Problems and their solutions :-

Q Find the expressions for the following operators :-

(i) $\left(\frac{d}{dx}+x\right)^2$

Let us consider a function $\psi(x)$ which is operated upon by the given operator, thus –

$$\left(\frac{d}{dx}+x\right)^2\psi(x) = \left(\frac{d}{dx}+x\right)\left(\frac{d}{dx}+x\right)\psi$$

$$= \left(\frac{d}{dx}+x\right)\left(\frac{d\psi}{dx}+x\cdot\psi\right)$$

$$= \left(\frac{d^2\psi}{dx^2}+2x\cdot\frac{d\psi}{dx}+x^2\psi+\psi\right)$$

$$= \left(\frac{d^2}{dx^2}+2x\cdot\frac{d}{dx}+x^2+1\right)\psi$$

$$\therefore \left(\frac{d}{dx}+x\right)^2 = \frac{d^2}{dx^2}+2x\cdot\frac{d}{dx}+x^2+1$$

Prove that :-

(ii) $\left(\frac{d}{dx}+\frac{1}{x}\right)^3 = \frac{d^3}{dx^3}+\frac{3}{x}\frac{d^2}{dx^2}$

(iii) $\left(\frac{d}{dx}x\right)^2 = \frac{x^2 d^2}{dx^2} + 3x\cdot\frac{d}{dx}+1$

(iv) $\left(x\cdot\frac{d}{dx}\right)^2 = x^2\frac{d^2}{dx^2}+x\cdot\frac{d}{dx}$

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Show that :-

$$\left[\hat{A}\,[\hat{B}\,\hat{C}]\right] = \left[[\hat{A}\,\hat{B}]\hat{C}\right] + \left[\hat{B}\,[\hat{A}\,\hat{C}]\right]$$

from L.H.S

$$[\hat{A}\,[\hat{B}\,\hat{C}]] = \hat{A}(\hat{B}\hat{C} - \hat{C}\hat{B}) - (\hat{B}\hat{C} - \hat{C}\hat{B})\hat{A}$$

$$= \hat{A}\hat{B}\hat{C} - \hat{A}\hat{C}\hat{B} - \hat{B}\hat{C}\hat{A} + \hat{C}\hat{B}\hat{A} \quad \text{--- (I)}$$

from R.H.S

$$[[\hat{A}\,\hat{B}]\,\hat{C}] + [\hat{B}\,[\hat{A}\,\hat{C}]]$$

$$= [(\hat{A}\hat{B} - \hat{B}\hat{A})\hat{C}] + [\hat{B}(\hat{A}\hat{C} - \hat{C}\hat{A})]$$

$$= \hat{A}\hat{B}\hat{C} - \cancel{\hat{B}\hat{A}\hat{C}} - \cancel{\hat{C}\hat{A}\hat{B}} + \hat{C}\hat{B}\hat{A} + \cancel{\hat{B}\hat{A}\hat{C}} - \hat{B}\hat{C}\hat{A} - \hat{A}\hat{C}\hat{B} + \cancel{\hat{C}\hat{A}\hat{B}}$$

$$= \hat{A}\hat{B}\hat{C} - \hat{A}\hat{C}\hat{B} - \hat{B}\hat{C}\hat{A} + \hat{C}\hat{B}\hat{A} \quad \text{--- (II)}$$

$\therefore$ LHS = R.H.S

Evaluate the Commutator

$$\left[x\ \frac{d}{dx}\right] = ?$$

By definition $\left[x \cdot \frac{d}{dx}\right] = x\frac{d}{dx} - \frac{d}{dx}x$

or $\left(x \cdot \frac{d}{dx} - \frac{d}{dx}x\right)\psi = \frac{x \cdot d\psi}{dx} - \frac{d}{dx}(x\psi)$

$$= \frac{x\,d\psi}{dx} - x\frac{d\psi}{dx} - \psi\frac{dx}{dx}$$

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$= -\psi$

Thus,

$$\left[x, \frac{d}{dx}\right] = -1$$

where, $\psi(x)$ is an arbitrary function.

Q. Prove that the quantum mechanical operators for the following observables are Hermitian (i) linear momentum (ii) angular momentum (iii) Energy.

Soln An operator $\hat{A}$ is said to be Hermitian if

$$\int \psi_i^* (\hat{A}\psi_j) d\tau = \int (\hat{A}\psi_i)^* \psi_j d\tau$$

where ψ_i & ψ_j are eigenfunctions of $\hat{A}$.

(i) The x-component of linear momentum operator p_x is $p_x = -i\hbar \frac{d}{dx}$

From the method of eigenfunction by parts

$$\int_{-\infty}^{\infty} \psi_i^* \left(-i\hbar \frac{d}{dx}\right) \psi\, dx = -i\hbar \left[\psi_i^* \psi_j\right]_{-\infty}^{\infty} + i\hbar \int_{-\infty}^{+\infty} \psi_j \frac{d\psi_i^*}{dx} dx$$

Since, ψ_i & ψ_j vanishes at infinity, so the first term on RHS must be zero.

$$\therefore \int_{-\infty}^{\infty} \psi_i^* \left(-i\hbar \frac{d}{dx}\right) \psi \, dx = \int_{-\infty}^{+\infty} \psi_j \left(-i\hbar \frac{d}{dx}\right)^* \psi_i^* dx$$

Hence, the operator for p_x is Hermitian.

(ii) Angular momentum$_x$

$$\hat{L}_x = -i\hbar\left(y\cdot\frac{d}{dz} - z\frac{d}{dy}\right)$$

for yourself (as in - (i))

and

(iii) Energy

$$\hat{H} = \hat{T} + \hat{V}$$

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& $$\hat{T} = \frac{-\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{dz^2}\right)$$

$$\hat{V} = -Ze^2/r$$

We have to show that

$$\int \Psi_i \hat{H} \Psi_j \, d\tau = \int \Psi_j \hat{H} \Psi_i \, d\tau \quad \text{——— (1)}$$

Putting the value of $\hat{H}$ by substituting it for $\hat{T}$ & $\hat{V}$ and obeying the postulates of quantum mechanics, On intigration by parts within $-\infty$ to $+\infty$, we get eqⁿ — (1)

∴ Hamiltonian operator is Hermitian.

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