Abstract Algebra

RINGS

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Definition :- Let R be a non-empty set. An algebraic structure $(R, +, \cdot)$ together with two binary operations addition and multiplication for all $a, b \in R$ is called a ring if this structure satisfies following properties:

(i) closed under addition:

$$a+b \in R \quad \forall\, a, b \in R$$

(ii) Associative under addition:

$$a+(b+c) = (a+b)+c, \quad \forall\, a, b, c \in R$$

(iii) Existence of identity:

$$0+a = a = a+0, \quad \forall\, a \in R$$

then 0 is an additive identity of R.

(iv) Existence of inverse:

there exists an element $-a \in R$ such that

$$-a+a = 0 = a+(-a), \quad \forall\, a \in R$$

then $-a$ is additive inverse of a.

(v) Commutative under addition:

$$a+b = b+a, \quad \forall\, a, b \in R$$

(vi) closed under multiplication:-

$$a \cdot b \in R \quad \forall\, a, b \in R$$

(vii) Associative under multiplication :-

$$a\cdot(b\cdot c) = (a\cdot b)\cdot c \quad \forall a,b,c \in R$$

$$a\cdot(b+c) = a\cdot b + a\cdot c \text{ (Left distributive)}$$

and $$(b+c)\cdot a = b\cdot a + c\cdot a \text{ (Right distributive)}$$

OR

An algebraic structure $(R,+,\cdot)$ is said to be a ring provided $a+b \in R$, $a\cdot b \in R$ for all $a,b \in R$ and satisfies following properties :

1. $(R, '+')$ is an abelian group
2. Multiplication is associative.
3. Multiplication is distributive over addition.
4. Multiplication need not be Commutative.

Ring with Unity :

Definition :- A ring R is said to be ring with unity if the multiplicative identity i.e. $1 \in R$ such that

$$1\cdot a = a = a\cdot 1 \quad \forall a \in R$$

Commutative Ring :

Definition :- A ring R is said to be a Commutative ring if

$$a\cdot b = b\cdot a, \quad \forall a,b \in R.$$

Some Examples of Rings :-

1. **Null ring (or zero ring)** :- A set R having a single element 0 with two binary operations addition and multiplication defined by $0+0=0$ and $0 \cdot 0 = 0$ is called a null ring.

2. **Ring of integers** :- The set Z of all integers with two binary operations to addition and multiplication forms a ring. This ring is called a ring of integers.

3. **Ring of real numbers** :- The set R of all real numbers with two binary operations addition and multiplication forms a ring which is called a ring of real numbers.

4. **Ring of rational number** :- The set Q of all rational number forms a commutative ring under addition and multiplication. This ring is called a ring of rational numbers.

5. **Ring of matrices** :- The set M of all matrices of order $n \times n$ whose elements as integers, real, complex number forms a non-commutative ring with unit elements under matrix addition and matrix multiplication is called a ring of matrices.