

# Abstract Algebra

①

Theorem :- If  $H$  be a normal subgroup of a group  $G$  and  $K$  is a normal subgroup of  $G$  containing  $H$ , then  $G/K \cong (G/H)/(K/H)$ .

Proof :- Since  $H$  is a normal subgroup of  $G$  and  $K$  is a normal subgroup of  $G$  containing  $H$ . (i.e.  $H \subseteq K$ )

So the quotient group  $K/H$  is a normal subgroup of the quotient group  $G/H$ .

Hence:  $(G/H)/(K/H)$  is a quotient group.

Now we consider a mapping:  $\phi: (G/H) \rightarrow (G/K)$  defined by:  $\phi(Hx) = Kx$ , where  $x \in G$ .

At first prove:  $\phi$  is well defined.

Let  $Hx = Hy$ ,  $x, y \in G$ .

$\therefore xy^{-1} \in H \Rightarrow xy^{-1} \in K$  since  $H \subseteq K$

$\Rightarrow Kx = Ky \Rightarrow \phi(Hx) = \phi(Hy)$

$\therefore Hx = Hy \Rightarrow \phi(Hx) = \phi(Hy)$ , so  $\phi$  is homomorphism.

Let  $x, y \in G$ .

Then  $\phi[(Hx) \cdot (Hy)] = \phi(Hxy) = Kxy$   
 $= (Kx)(Ky)$   
 $= \phi(Hx) \phi(Hy)$

So  $\phi$  is homo.

Again: for any  $Kx \in G/K$ , there must

exists  $Hx \in G/H$   
 such that  $\phi(Hx) = Kx$   
 So  $\phi$  is onto also.

The identity elements of  $G/K$  is  $K$ .

if  $Hx \in G/H$   
 then  $Hx \in \text{Kernel of } \phi \Leftrightarrow \phi(Hx) = K$   
 $\Leftrightarrow Kx = K$   
 $\Leftrightarrow x \in K$   
 $\Leftrightarrow Hx \in K/H$

Hence Kernel of  $\phi = K/H$ , which is a subset of  $G/H$

So  $\phi$  is a homomorphism of  $G/H$  onto  $G/K$  with Kernel  $K/H$ . Therefore by fundamental theorem of homomorphism of groups, we have

$$G/K \cong (G/H) / (K/H)$$

Proved

### Solvable Groups :-

A group  $G$  is said to be solvable if we can find a finite chain of subgroups.

$$G = N_0 \supseteq N_1 \supseteq N_2 \supseteq \dots \supseteq N_k = (e).$$

such that each  $N_i$  is a normal subgroup of  $N_{i-1}$  and each quotient group  $N_{i-1}/N_i$  is abelian.

Normal series of a group:-

A finite sequence of subgroups

$$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \dots \supseteq G_k = (e)$$

of a group  $G$  is called a subnormal series of  $G$  if  $G_{i-1}$  is a normal subgroup of  $G_i$ ,  $\forall i = 0, 1, 2, \dots, k-1$ .

The quotient groups  $G_i/G_{i+1}$  are called the factor groups of the subnormal series. If each  $G_i$  is a normal subgroup of  $G$  itself, then the series is said to be a normal series of  $G$ .

Q.) Prove that every abelian group is solvable.  
sol: Let  $G$  is an abelian group. Let  $G_0 = N_0$  and  $N_1 = (e)$

Then  $G = N_0 \supseteq N_1 = (e)$  is a solvable series of  $G$ .

Clearly  $N_1 = (e)$  is a normal subgroup of  $N_0 = G$ . Since for any  $a \in G$  we have

$$aea^{-1} = a^{-1}a = e \in (e) = N_1$$

Since  $G$  is abelian, so the quotient group  $G/N_1 = G/(e) = G$  is abelian.

As every quotient group of an abelian group is abelian. So  $G$  is solvable.

Proved