

Abstract Algebra

General properties of Groups

Theorem 3:- If a, b, c are three elements of a group $(G, *)$ then

$$(a * c) = b * c \Rightarrow a = b \quad (\text{Right Cancellation law})$$

$$c * a = c * b \Rightarrow a = b \quad (\text{Left Cancellation law})$$

Proof:- (i) $a * c = b * c \Rightarrow (a * c) * c^{-1} = (b * c) * c^{-1} \quad (\because c^{-1} \in G)$

$$\Rightarrow a * (c * c^{-1}) = b * (c * c^{-1})$$

(By associativity)

$$\Rightarrow a * e = b * e \Rightarrow a = b$$

$$c * a = c * b \Rightarrow c^{-1} * c * a = c^{-1} * c * b$$

$$\Rightarrow (c^{-1} * c) * a = (c^{-1} * c) * b$$

$$\Rightarrow e * a = e * b \Rightarrow a = b$$

Theorem 4:- In a group G , the equations $a * x = b$ and $y * a = b$ have solutions in G where $a, b \in G$.

Proof:-

$$a * x = b$$

$$\Rightarrow a^{-1} * (a * x) = a^{-1} * b \quad (\because a^{-1} \in G)$$

$$\Rightarrow (a^{-1} * a) * x = a^{-1} * b \quad (\text{by associativity})$$

$$\Rightarrow e * x = a^{-1} * b$$

$$\Rightarrow x = a^{-1} * b \in G$$

Therefore, the equation $a * x = b$ has a solution $x = a^{-1} * b$ in G . Similarly, the equation $y * a = b$ has a solution $y = b * a^{-1}$ in G .

Uniqueness :-

Let, if possible x_1 and x_2 be any two solutions of the equation $a * x = b$ so that

$$a * x_1 = b \quad \text{and} \quad a * x_2 = b$$

$$\Rightarrow a * x_1 = a * x_2 \Rightarrow x_1 = x_2$$

(By left cancellation law)

Therefore, the solution of the equation $a * x = b$ is unique. Similarly, it can be proved that the equation $x * a = b$ has a unique solution.

Hence, the given equation have unique solutions in G .

Remark

① with the help of the above theorem, we can define the group alternatively as follows:

"A set G with a binary composition $*$ is a group iff"

- (i) the composition $*$ is associative
- (ii) the equations $ax = b$ and $ya = b$ have unique solutions in G .